

THE VISUAL ANGLE METRIC AND MÖBIUS TRANSFORMATIONS

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ABSTRACT. A new similarity-invariant metric v_G , defined on a domain $G \subsetneq \mathbb{R}^n$ whose boundary is not a proper subset of a line, is introduced. We find sharp bounds for v_G in terms of the hyperbolic metric in the unit ball and the upper half space. We also obtain sharp Lipschitz constants w. r. t. v_G under the Möbius transformations from the upper half plane onto itself and onto the unit disk as well as from the unit ball onto itself.

KEYWORDS. the visual angle metric, the hyperbolic metric, Lipschitz constant

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1. INTRODUCTION

As the name suggests, metrics play a crucial role in geometric function theory of the plane. The three classical geometries, the Euclidean, the hyperbolic, and the spherical geometry each has a metric, the Euclidean, the hyperbolic and the chordal metric, resp., that is invariant under a group of Möbius transformations. These three groups of transformations are isometric automorphisms of the respective spaces, the complex plane \mathbb{C} , the unit disk, and the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

In addition to these classical geometries, many novel ways to look at geometric function theory have been introduced recently, in the study of hyperbolic type geometries. For instance, the Apollonian metric, the Möbius invariant metric, the quasihyperbolic metric and some weak metrics have been studied in [Be2, AST, S, HIMPS, KL, L2, PT]. Furthermore, metrics based on conformal capacity have been studied in [S]. On one hand, these metrics share several properties of hyperbolic metrics and are therefore sometimes called hyperbolic type metrics. On the other hand, these metrics differ from the classical hyperbolic metric in other respects. This circumstance suggests a wide spectrum of open problems concerning the geometry of these metric spaces and homeomorphisms between two such spaces. Several open problems of this character were listed in [Vu2]. While some of these problems have been studied and solved [K, KVZ, L2, S], the systematic study of these problems is still in its initial stages.

The metrics introduced in [AST] offer a way to connect the behavior of the basic notion of an angle to the behavior of a map. Classically one studies angle distortion locally, "in the small", whereas in [AST] this topic is studied "in the large". Here we give an alternative way to look at this topic and introduce here what we call the visual angle metric*. We begin by the formulation of some of our main results. For the definitions, the reader is referred to Section 2.

Theorem 1.1. *For $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$ and $x, y \in G$, let $\rho_G^*(x, y) = \arctan(\operatorname{sh} \frac{\rho_G(x, y)}{2})$. Then*

$$\rho_G^*(x, y) \leq v_G(x, y) \leq 2\rho_G^*(x, y).$$

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*The term "visual metric" occurs in a different meaning in the study of Gromov hyperbolic spaces, see [BS, section 3.3].

The left-hand side of the inequality is sharp and the constant 2 in the right-hand side of the inequality is best possible.

Theorem 1.2. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$ be a Möbius transformation. Then*

$$\sup_{\substack{f \in \mathcal{GM}(\mathbb{B}^n), \\ x \neq y \in \mathbb{B}^n}} \frac{v_{\mathbb{B}^n}(f(x), f(y))}{v_{\mathbb{B}^n}(x, y)} = 2.$$

Theorem 1.3. *Let $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be a Möbius transformation. Then for all $x, y \in \mathbb{H}^2$*

$$v_{\mathbb{H}^2}(x, y)/2 \leq v_{\mathbb{B}^2}(f(x), f(y)) \leq 2v_{\mathbb{H}^2}(x, y),$$

and the constants $1/2$ and 2 are both best possible.

Theorem 1.4. *Let $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ and $c \neq 0$. Let $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be a Möbius transformation with $f(z) = \frac{az+b}{cz+d}$. Then*

$$\sup_{x \neq y \in \mathbb{H}^2} \frac{v_{\mathbb{H}^2}(f(x), f(y))}{v_{\mathbb{H}^2}(x, y)} = 2.$$

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. Notation. Throughout this paper we will discuss domains $G \subset \mathbb{R}^n$, i.e., open and connected subsets of \mathbb{R}^n . For $x, y \in G$ the Euclidean distance between x and y is denoted by $|x - y|$ or $d(x, y)$, as usual. The notation $d(x, \partial G)$ stands for the distance from the point x to the boundary ∂G of the domain G . The Euclidean n -dimensional ball with center z and radius r is denoted by $\mathbb{B}^n(z, r)$, and its boundary sphere by $S^{n-1}(z, r)$. In particular, $\mathbb{B}^n(r) = \mathbb{B}^n(0, r)$, $S^{n-1}(r) = S^{n-1}(0, r)$, and $\mathbb{B}^n = \mathbb{B}^n(0, 1)$, $S^{n-1} = S^{n-1}(0, 1)$. The upper Lobachevsky n -dimensional half space (as a set) is denoted by $\mathbb{H}^n = \{(z_1, z_2, \dots, z_n) \in \mathbb{R}^n : z_n > 0\}$. For $t \in \mathbb{R}$ and $a \in \mathbb{R}^n \setminus \{0\}$ we denote a hyperplane in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ by $P(a, t) = \{x \in \mathbb{R}^n : x \cdot a = t\} \cup \{\infty\}$.

Given two points x and y the segment between them is denoted by

$$[x, y] = \{(1 - t)x + ty : 0 \leq t \leq 1\}.$$

Given a vector $u \in \mathbb{R}^n \setminus \{0\}$ and a point $x \in \mathbb{R}^n$, the line passing through x with direction vector u is denoted by $L(x, u)$. The open ray emanating from x in the direction of u is denoted by $\text{ray}(x, u)$. The hyperplane orthogonal to u and passing through x is denoted by $P_x(u)$. Hence

$$\begin{aligned} L(x, u) &= \{x + tu : t \in \mathbb{R}\}, \\ \text{ray}(x, u) &= \{x + tu : t > 0\}, \\ P_x(u) &= P(u, x \cdot u). \end{aligned}$$

Given three distinct points x, y and $z \in \mathbb{R}^n$, the notation $\angle(x, z, y)$ means the angle in the range $[0, \pi]$ between the segments $[x, z]$ and $[y, z]$.

2.2. Möbius transformations. The group of Möbius transformations in $\overline{\mathbb{R}^n}$ is generated by transformations of two types:

(1) reflections in the hyperplane $P(a, t)$:

$$f_1(x) = x - 2(x \cdot a - t) \frac{a}{|a|^2}, \quad f_1(\infty) = \infty;$$

(2) inversions (reflections) in the sphere $S^{n-1}(a, r)$:

$$f_2(x) = a + \frac{r^2(x-a)}{|x-a|^2}, \quad f_2(a) = \infty, f_2(\infty) = a.$$

If $G \subset \overline{\mathbb{R}^n}$ we denote by $\mathcal{GM}(G)$ the group of all Möbius transformations with $fG = G$.

We denote $a^* = \frac{a}{|a|^2}$ for $a \in \mathbb{R}^n \setminus \{0\}$, and $0^* = \infty$, $\infty^* = 0$. For a fixed $a \in \mathbb{B}^n \setminus \{0\}$, let

$$\sigma_a(z) = a^* + r^2(x - a^*)^*, \quad r^2 = |a|^{-2} - 1$$

be the inversion in the sphere $S^{n-1}(a^*, r)$ orthogonal to S^{n-1} . Then $\sigma_a(a) = 0$, $\sigma_a(a^*) = \infty$.

Let p_a denote the reflection in the $(n-1)$ -dimensional hyperplane $P(0, a)$ and define a sense-preserving Möbius transformation by

$$(2.3) \quad T_a = p_a \circ \sigma_a.$$

Then, $T_a \mathbb{B}^n = \mathbb{B}^n$, $T_a(a) = 0$, and $T_a(e_a) = e_a$, $T_a(-e_a) = -e_a$. For $a = 0$ we set $T_0 = id$, where id stands for the identity map. It is easy to see that $(T_a)^{-1} = T_{-a}$. It is well-known that there is an orthogonal map k such that $g = k \circ T_a$ if $g \in \mathcal{GM}(\mathbb{B}^n)$, where $a = g^{-1}(0)$ [Be1, p.40, Theorem 3.5.1].

2.4. Lipschitz mappings. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \rightarrow Y$ be continuous and let $L \geq 1$. We say that f is L -lipschitz, if

$$d_Y(f(x), f(y)) \leq L d_X(x, y), \quad \text{for } x, y \in X,$$

and L -bilipschitz, if f is a homeomorphism and

$$d_X(x, y)/L \leq d_Y(f(x), f(y)) \leq L d_X(x, y), \quad \text{for } x, y \in X.$$

A 1-bilipschitz mapping is called an isometry.

2.5. Absolute Ratio. For an ordered quadruple a, b, c, d of distinct points in $\overline{\mathbb{R}^n}$ we define the absolute (cross) ratio by

$$|a, b, c, d| = \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)},$$

where $q(x, y)$ is the spherical (chordal) metric, defined by

$$\begin{cases} q(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x, y \neq \infty, \\ q(x, \infty) = \frac{1}{\sqrt{1+|x|^2}}, & x \neq \infty. \end{cases}$$

Note also that for distinct points $a, b, c, d \in \mathbb{R}^n$

$$|a, b, c, d| = \frac{|a-c||b-d|}{|a-b||c-d|}.$$

The most important property of the absolute ratio is the Möbius invariance, see [Be1, p.32, Theorem 3.2.7], i.e., if f is a Möbius transformation, then

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|,$$

for all distinct $a, b, c, d \in \overline{\mathbb{R}^n}$.

2.6. Hyperbolic metric. By [Be1, p.35] we have

$$(2.7) \quad \text{ch}\rho_{\mathbb{H}^n}(x, y) = 1 + \frac{|x-y|^2}{2d(x, \partial\mathbb{H}^n)d(y, \partial\mathbb{H}^n)}$$

for all $x, y \in \mathbb{H}^n$ and by [Be1, p.40] we have

$$(2.8) \quad \operatorname{sh} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}$$

for all $x, y \in \mathbb{B}^n$. In particular, for $t \in (0, 1)$,

$$(2.9) \quad \rho_{\mathbb{B}^2}(0, te_1) = \log \frac{1+t}{1-t} = 2 \operatorname{arth} t.$$

The hyperbolic metric is invariant under Möbius transformations.

2.10. Distance ratio metric. For a proper subdomain $G \subset \mathbb{R}^n$ and for all $x, y \in G$ the distance ratio metric j_G is defined as

$$j_G(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}} \right).$$

The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [GP] and in the above simplified form by M. Vuorinen [Vu1] and it is frequently used in the study of hyperbolic type metrics [HIMPS] and geometric theory of functions.

2.11. Quasihyperbolic metric. Let G be a proper subdomain of \mathbb{R}^n . For all $x, y \in G$, the quasihyperbolic metric k_G is defined as

$$k_G(x, y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z, \partial G)} |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G [GP].

2.12. Ptolemaic angular metrics. We start by defining the two so called Ptolemaic metrics, the first of which has recently been studied in [AST]. The other one is a one-boundary-point version of the same metric.

We call a metric space (X, m) a *Ptolemaic space* if it satisfies the *Ptolemy inequality*

$$m(x, z)m(y, w) \leq m(x, w)m(y, z) + m(x, y)m(w, z)$$

for all $x, y, z, w \in X$. In general metric spaces need not be Ptolemaic, but for instance all inner product spaces are, and in particular the Euclidean space is Ptolemaic. Also note that if we choose for instance $w = \infty$, then Ptolemy's inequality in the Euclidean space reduces to the triangle inequality

$$|x - z| \leq |y - z| + |x - y|.$$

Now define the *angular characteristic* of four points a, b, c, d by

$$\sigma(a, b, c, d) = \frac{|a - c||b - d|}{|a - b||c - d| + |a - d||b - c|}.$$

V. Aseev, A. Sychëv and A. Tetenov proved in [AST, Lemma 1.6], that for a given Ptolemaic space X , nonempty sets $A, B \subset X$ for which $\operatorname{card}(A \cup B) \geq 2$ and $A \cup B \neq X$, the function

$$r_{AB}(x, y) = \sup_{a \in A, b \in B, a \neq b} \sigma(a, x, b, y)$$

is a metric on $X \setminus (A \cup B)$. They call this the *angular metric*, and also prove its Möbius invariance in [AST, Theorem 2.4]. In this article we consider also the word angular metrics in different meanings. However, applying the work of Aseev and his collaborators to the case $X = \overline{G}$, $A = B = \partial G$, where $G \subsetneq \mathbb{R}^n$ is a domain, we obtain the following definition:

Definition 2.13. Given $G \subsetneq \mathbb{R}^n$ and $x, y \in G$, we define a Möbius invariant metric by

$$r_G(x, y) = \sup_{z, w \in \partial G, z \neq w} \sigma(z, x, w, y) = \sup_{z, w \in \partial G, z \neq w} \frac{|z - w||x - y|}{|z - x||w - y| + |z - y||w - x|},$$

and call this the *Ptolemaic angular metric*.

As the reader may have noticed, for instance the Apollonian and half-apollonian metrics are actually defined in the same way, only in the half-apollonian case one of the boundary points in the definition is “forced” to infinity. This approach is used in other related metrics as well, such as the frequently used distance ratio metrics j and \hat{j} by M. Vuorinen and F. Gehring, respectively (see [Vu1] for properties of these metrics). Lately, A. Papadopoulos and M. Troyanov have described this as such metrics being two different symmetrizations, the max-symmetrization and the mean value-symmetrization, of the same weak metric, see [PT].

We next develop the one-point version of the Ptolemaic angular metric. This was also briefly considered in [AST, p.192] and [H, Lemma 6.1]. However, to our knowledge this particular metric has not been studied to any further extent.

Definition 2.14. Given $G \subsetneq \mathbb{R}^n$ and $x, y \in G$, we define a similarity invariant metric by

$$s_G(x, y) = \sup_{z \in \partial G} \frac{|x - y|}{|z - x| + |z - y|} \in [0, 1],$$

and call this the *triangular ratio metric*.

The mutual order between the two metrics s_G and r_G is a direct consequence of the definitions.

Proposition 2.15. Let $G \subsetneq \mathbb{R}^n$ and $\infty \in \partial G$, then for all $x, y \in G$

$$s_G(x, y) \leq r_G(x, y).$$

Proof. By Definition 2.13 and Definition 2.14, we get

$$s_G(x, y) = \sup_{z \in \partial G} \sigma(z, x, \infty, y) \leq \sup_{z, w \in \partial G} \sigma(z, x, w, y) = r_G(x, y).$$

□

Remark 2.16. Even if perhaps the most natural way to define the triangular ratio metric is the direct formula given in Definition 2.14, it is also possible to give a definition similar to the one for the visual angle metric as follows. Namely, given two distinct points $x, y \in \mathbb{R}^n$, and $c > 0$ the set

$$\{z \in \mathbb{R}^n : |x - z| + |y - z| = c\}$$

is known to be an ellipsoid with x and y as foci, semimajor axis $c/2$ and $n - 1$ semiminor axes $\sqrt{c^2 - |x - y|^2}/2$. Denoting the ellipsoid with foci x and y , semimajor axis b and $n - 1$ semiminor axes a by $F(x, y; a, b)$, we may define the c -envelope of the pair (x, y) as

$$F_{xy}^c = [x, y] \cup \left(\bigcup_{|x-y| < t \leq c} F(x, y; \tfrac{1}{2}\sqrt{t^2 - |x-y|^2}, t/2) \right).$$

Then it is easy to see that the above set has an alternative definition, namely

$$F_{xy}^c = \{z \in \mathbb{R}^n : |x - z| + |y - z| \leq c\}.$$

Therefore, we can define the triangular ratio metric in another way.

Remark 2.17. For all $x, y \in G$, the metric s_G is defined by

$$s_G(x, y) = \frac{|x - y|}{\inf\{c : F_{xy}^c \cap \partial G \neq \emptyset\}}.$$

This geometric approach to the triangular ratio metric gives a convenient tool for instance to compare it with the visual angle metric, as will be seen later.

2.18. Visual angle metric. We introduce two versions of angle metrics, a "one-point version" corresponding to a max-argument and a "two-point version" corresponding to a mean-value argument. The one-point version of this metric was introduced for dimension $n = 2$ in [L1], there also under the name "angular metric", and the notation ω_G . The definition is unfortunately rather involved in higher dimensions, but the intuition is fairly easy.

We begin by introducing some geometric concepts and notation. Let $x, y \in \mathbb{R}^n$, $x \neq y$ and $0 < \alpha < \pi$. Let $m = (x + y)/2$ be the midpoint of the segment $[x, y]$ and define $P_{xy} = P_m(x - y)$. Furthermore, let $S(z, r) \subset \mathbb{R}^n$ be a circle centered at z with radius $r = |x - y|/(2 \sin \alpha)$ such that $x, y \in S(z, r)$. Clearly the center z is a point in P_{xy} .

Now denote

$$\mathcal{C}_{xy}^\alpha = \left\{ S(z, r) : z \in P_{xy}, d(z, [x, y]) = \frac{|x - y|}{2 \tan \alpha}, r = \frac{|x - y|}{2 \sin \alpha} \right\}.$$

Every circle $C \in \mathcal{C}_{xy}^\alpha$ contains the points x and y and therefore $C \setminus \{x, y\}$ consists of two circular arcs. We denote these two circular arcs by $\text{comp}_\alpha(C)$ and $\text{comp}_{\pi-\alpha}(C)$ and assume that the length of $\text{comp}_\alpha(C)$ is equal to $2(\pi - \alpha)|x - z|$, see Figure 1. Then it is clear that

$$C = \{x\} \cup \{y\} \cup \text{comp}_\alpha(C) \cup \text{comp}_{\pi-\alpha}(C).$$

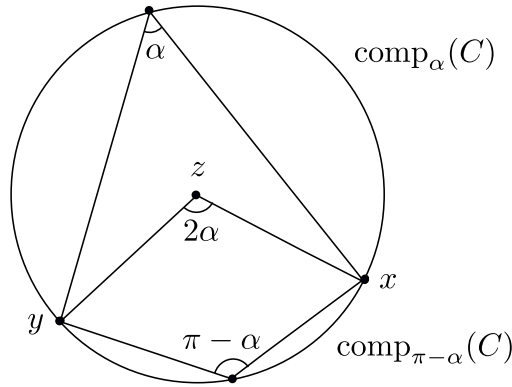


FIGURE 1. Components $\text{comp}_\alpha(C)$ and $\text{comp}_{\pi-\alpha}(C)$ of the circle C .

Finally, we define the α -envelope of the pair (x, y) to be

$$E_{xy}^\alpha = [x, y] \cup \left(\bigcup_{C \in \mathcal{C}_{xy}^\alpha, \alpha \leq t < \pi} \text{comp}_\alpha(C) \right)$$

if $0 < \alpha < \pi$, $E_{xy}^0 = \mathbb{R}^n$ and $E_{xy}^\pi = [x, y]$. For instance in the case $n = 3$, this means that for $0 < \alpha < \pi/2$ the set E_{xy}^α is an "apple domain", for $\alpha = \pi/2$ the ball $\mathbb{B}^3(m, |x - y|/2)$, and for $\pi/2 < \alpha < \pi$ a "lemon domain".

The set E_{xy}^α has the property that for all $w \in \partial E_{xy}^\alpha$ the angle $\angle(x, w, y)$ equals to α .

Remark 2.19. It is not difficult to show that in fact

$$E_{xy}^\alpha = \{z \in \mathbb{R}^n : \angle(x, z, y) \geq \alpha\}.$$

It is easy to see that (Figure 2)

$$(2.20) \quad E_{xy}^{2 \arcsin \frac{|x-y|}{c}} \subset F_{xy}^c \quad \text{and} \quad E_{xy}^\alpha \subset F_{xy}^{|x-y|/\sin \frac{\alpha}{2}}.$$

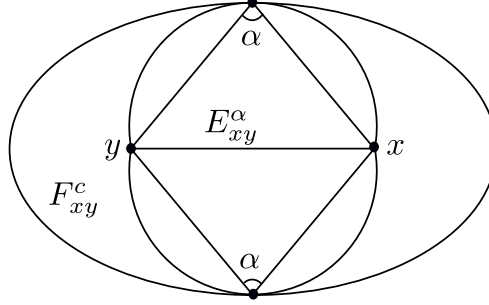


FIGURE 2. Here $\alpha = 2 \arcsin \frac{|x-y|}{c}$ or $c = |x-y|/\sin \frac{\alpha}{2}$.

Now we are ready for the following definition:

Definition 2.21. Let $G \subsetneq \mathbb{R}^n$ be a domain and $x, y \in G$. We define a distance function v_G by

$$v_G(x, y) = \sup \{ \alpha : E_{xy}^\alpha \cap \partial G \neq \emptyset \}.$$

Next we show that the function v_G in fact defines a similarity invariant metric in all domains where it is defined.

Lemma 2.22. The function $v_G: G \times G \rightarrow [0, \pi]$ is a similarity invariant pseudometric for every domain $G \subsetneq \mathbb{R}^n$. It is a metric unless ∂G is a proper subset of a line and will be called the visual angle metric.

Proof. Let $x, y \in G$ be arbitrary. Clearly $\angle(x, z, y) = 0$ if and only if $x = y$ or z is located on $\text{ray}(x, x-y)$ or $\text{ray}(y, y-x)$. Thus $v_G(x, y) = 0$ implies $x = y$, unless ∂G is a proper subset of a line.

We now prove the triangle inequality for v_G . Let $x, y, z \in G$. Let w be an arbitrary point in $E_{xy}^\alpha \cap \partial G$, where E_{xy}^α is the envelope such that α is the supremum angle in Definition 2.21. Without loss of generality, we can assume that x, y, z, w are in \mathbb{R}^3 . Let $r = \min\{|w-x|, |w-y|, |w-z|\}/2$, and the points x', y' and z' denote the intersections of $S^2(w, r)$ with $[w, x]$, $[w, y]$ and $[w, z]$, respectively. Clearly

$$\angle(x, w, y) = \angle(x', w, y'), \quad \angle(z, w, y) = \angle(z', w, y'), \quad \angle(z, w, x) = \angle(z', w, x').$$

Also, by considering the intrinsic metric of the sphere $S^2(w, r)$ (see [B, p.287, 18.6.10]), it is clear that

$$(2.23) \quad \angle(x', w, y') \leq \angle(x', w, z') + \angle(z', w, y').$$

Let $\beta = \angle(x, w, z)$ and $\gamma = \angle(z, w, y)$. By the definition of v_G it is clear that $\beta \leq v_G(x, z)$ and $\gamma \leq v_G(z, y)$. From this, and the inequality (2.23) it now follows that

$$\begin{aligned} v_G(x, y) &= \alpha = \angle(x, w, y) = \angle(x', w, y') \leq \angle(x', w, z') + \angle(z', w, y') \\ &= \angle(x, w, z) + \angle(z, w, y) \leq v_G(x, z) + v_G(z, y). \end{aligned}$$

This proves the triangle inequality. Similarity invariance is clear, as the shape of envelopes are similarity invariant. \square

It follows from Remark 2.17 and Definition 2.21 that if G_1, G_2 are proper subdomains of \mathbb{R}^n , $G_1 \subset G_2$, and $x, y \in G_1$ are distinct points, then $s_{G_1}(x, y) \geq s_{G_2}(x, y)$ and $v_{G_1}(x, y) \geq v_{G_2}(x, y)$. It is also evident that for $x, y \in G$ if there exists $z \in [x, y]$ and $z \notin G$ then $v_G(x, y) = \pi$.

2.24. Möbius-invariant version of the visual angle metric. By Definition 2.21, Remark 2.19 and the law of cosines it is immediately clear that the metric v_G also has the representation

$$(2.25) \quad \begin{aligned} v_G(x, y) &= \sup_{z \in \partial G} \arccos \frac{1}{2} \left(\frac{|x - z|}{|y - z|} + \frac{|y - z|}{|x - z|} - \frac{|x - y|^2}{|x - z||y - z|} \right) \\ &= \sup_{z \in \partial G} \arccos \frac{1}{2} (|z, y, x, \infty| + |z, x, y, \infty| - s(z, x, y, \infty)), \end{aligned}$$

where $s(a, b, c, d) = |a, b, d, c|/|a, c, d, b|$ is the *symmetric ratio*. For properties of the symmetric ratio, see for instance [Vu1, p.38–39]. From this representation it is also easy to immediately verify the similarity invariance. Next we construct a Möbius-invariant version of the visual angle metric.

Lemma 2.26. *The function $\bar{v}_G: G \times G \rightarrow [0, \pi]$ defined by*

$$(2.27) \quad \begin{aligned} \bar{v}_G(x, y) &= \sup_{z, w \in \partial G} \arccos \frac{1}{2} (|z, y, x, w| + |z, x, y, w| - s(z, x, y, w)) \\ &= \sup_{z, w \in \partial G} \arccos \frac{1}{2} \left(\frac{|x - z||y - w|}{|y - z||x - w|} + \frac{|y - z||x - w|}{|x - z||y - w|} \right. \\ &\quad \left. - \frac{|x - y|^2|z - w|^2}{|x - z||x - w||y - z||y - w|} \right) \end{aligned}$$

is a Möbius invariant pseudometric for every domain $G \subsetneq \mathbb{R}^n$. It is a metric whenever ∂G is not a proper subset of a line or a circle and will be called the visual double angle metric.

Proof. Möbius invariance is immediate by Möbius invariance of the absolute ratio. For the triangle inequality, let $a, b \in \partial G$, and $f: \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$ be a Möbius transformation such that $f(b) = \infty$. Then, for $x, y, z \in G$ we get

$$\begin{aligned} &\arccos \frac{1}{2} (|a, y, x, b| + |a, x, y, b| - s(a, x, y, b)) \\ &= \arccos \frac{1}{2} (|f(a), f(y), f(x), \infty| + |f(a), f(x), f(y), \infty| - s(f(a), f(x), f(y), \infty)) \\ &= \angle(f(x), f(a), f(y)). \end{aligned}$$

As in the proof of Lemma 2.22, we see that in the domain fG the inequality

$$\angle(f(x), f(a), f(y)) \leq \angle(f(x), f(a), f(z)) + \angle(f(z), f(a), f(y))$$

holds, so we get

$$\begin{aligned}
& \arccos \frac{1}{2} (|a, y, x, b| + |a, x, y, b| - s(a, x, y, b)) \\
& \leq \angle(f(x), f(a), f(z)) + \angle(f(z), f(a), f(y)) \\
& = \arccos \frac{1}{2} (|f(a), f(z), f(x), \infty| + |f(a), f(x), f(z), \infty| - s(f(a), f(x), f(z), \infty)) \\
& \quad + \arccos \frac{1}{2} (|f(a), f(z), f(y), \infty| + |f(a), f(y), f(z), \infty| - s(f(a), f(y), f(z), \infty)) \\
& = \arccos \frac{1}{2} (|a, z, x, b| + |a, x, z, b| - s(a, x, z, b)) \\
& \quad + \arccos \frac{1}{2} (|a, z, y, b| + |a, y, z, b| - s(a, y, z, b)) \leq \bar{v}_G(x, z) + \bar{v}_G(y, z),
\end{aligned}$$

which proves the triangle inequality as the above holds for all points $b \in \partial G$. The symmetricity and reflexivity are clear. It is also easy to show, that when $f(x) \neq f(y)$, $\angle(f(x), f(a), f(y))$ is zero exactly for point $f(a)$ on $\text{ray}(f(x), f(x) - f(y))$ or $\text{ray}(f(y), f(y) - f(x))$. Thus the statement follows by the circle preserving property of Möbius transformations. \square

As for the two Ptolemaic angular metrics, the visual angular metric and the visual double angle metric satisfy an obvious mutual ordering, which is proved exactly like Proposition 2.15, using the formulas (2.25) and (2.27).

Proposition 2.28. Let $G \subsetneq \mathbb{R}^n$ and $\infty \in \partial G$, then for all $x, y \in G$

$$v_G(x, y) \leq \bar{v}_G(x, y).$$

The following lemma is useful in verifying the triangle inequality.

Lemma 2.29. [Vu1, p.40, Exercise 3.33] Let $f : [0, \infty) \rightarrow [0, \infty)$ be increasing with $f(0) = 0$ such that $f(t)/t$ is decreasing on $(0, \infty)$. Then for all $s, t \geq 0$

$$f(s + t) \leq f(s) + f(t).$$

Sometimes it might be more convenient to study the metrics if the inconvenient inverse cosine function is removed from the definition.

Corollary 2.30. The functions v_G^* and \bar{v}_G^* from $G \times G$ onto $[0, 1]$, defined for all $x, y \in G \subsetneq \mathbb{R}^n$ by

$$v_G^*(x, y) = \sin \left(\frac{v_G(x, y)}{2} \right) \quad \text{and} \quad \bar{v}_G^*(x, y) = \sin \left(\frac{\bar{v}_G(x, y)}{2} \right).$$

Then v_G^* is a similarity-invariant metric provided ∂G is not a proper subset of a line and \bar{v}_G^* is a Möbius-invariant metric provided ∂G is not a proper subset of a line or a circle. Moreover, for all $x, y \in G$, there hold

$$v_G(x, y)/\pi \leq v_G^*(x, y) \leq v_G(x, y)/2 \quad \text{and} \quad \bar{v}_G(x, y)/\pi \leq \bar{v}_G^*(x, y) \leq \bar{v}_G(x, y)/2.$$

Proof. The function $f : x \mapsto \sin(x/2)$ is increasing on $[0, \pi]$, $f(x)/x$ is decreasing on $(0, \pi)$, and $f(0) = 0$. By Lemma 2.29 the triangle inequality follows and hence v_G^* and \bar{v}_G^* are metrics. The inequalities follow from the inequality $2x/\pi \leq \sin x \leq x$, valid in the interval $x \in [0, \pi/2]$. \square

Theorem 2.31. For all $G \subsetneq \mathbb{R}^n$ and all points $x, y \in G$ the inequalities

$$v_G^*(x, y) \leq s_G(x, y) \quad \text{and} \quad \bar{v}_G^*(x, y) \leq r_G(x, y)$$

hold.

Proof. By (2.20) we have

$$\begin{aligned} E_{xy}^\alpha &= \{z \in \mathbb{R}^n : \angle(x, z, y) \geq \alpha\} \\ &\subseteq F_{xy}^{|x-y|/\sin \frac{\alpha}{2}} = \left\{z \in \mathbb{R}^n : \frac{|x-y|}{|x-z|+|z-y|} \geq \sin(\alpha/2)\right\}. \end{aligned}$$

Thus $v_G^*(x, y) \leq s_G(x, y)$, and we see that equality holds if the boundary point $z \in E_{xy}^\alpha \cap F_{xy}^c$ such that both the angle α and the semimajor axis $c/2$ attain the supremum w.r.t. the domain G in Definition 2.21 and Remark 2.17, respectively.

For the metric \bar{v}_G^* a technique similar to the approach in Lemma 2.26 will be used. Let $a, b \in \partial G$, and let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Möbius mapping such that $f(b) = \infty$. As in the proof of Lemma 2.26 we see that

$$\begin{aligned} &\arccos \frac{1}{2} (|a, y, x, b| + |a, x, y, b| - s(a, x, y, b)) \\ &= \angle(f(x), f(a), f(y)) \leq \sup_{z \in \partial fG} \angle(f(x), z, f(y)). \end{aligned}$$

Since the function $g(x) = \sin(x/2)$ is increasing on $[0, \pi]$, we get

$$\begin{aligned} &\sin \left(\frac{1}{2} \arccos \frac{1}{2} (|a, y, x, b| + |a, x, y, b| - s(a, x, y, b)) \right) \\ &\leq \sup_{z \in \partial fG} \sin \left(\frac{\angle(f(x), z, f(y))}{2} \right) = v_{fG}^*(f(x), f(y)). \end{aligned}$$

Using the first inequality in this theorem and Proposition 2.15, we see that

$$\begin{aligned} &\sin \left(\frac{1}{2} \arccos \frac{1}{2} (|a, y, x, b| + |a, x, y, b| - s(a, x, y, b)) \right) \\ &\leq s_{fG}(f(x), f(y)) \leq r_{fG}(f(x), f(y)) = r_G(x, y). \end{aligned}$$

and the statement follows as b is chosen arbitrarily. \square

3. THE VISUAL ANGLE METRIC IN SOME SIMPLE DOMAINS

In this section, we consider the visual angle metric in some simple domains.

3.1. The punctured space $G_1 = \mathbb{R}^n \setminus \{0\}$. For $x, y \in G_1$, we have

$$v_{G_1}(x, y) = \angle(x, 0, y) \in [0, \pi]$$

and it is easy to see that v_{G_1} is only a pseudometric.

H. Lindén derived the sharp uniformity constant for G_1 in [L2, Theorem 1.6], i.e., for all $x, y \in G_1$

$$k_{G_1}(x, y) \leq \frac{\pi}{\log 3} j_{G_1}(x, y).$$

In G_1 the quasihyperbolic metric k and the distance ratio metric j are connected to v as follows

$$v_{G_1}(x, y) = \sqrt{k_{G_1}^2(x, y) - \log^2 \frac{|y|}{|x|}} \leq k_{G_1}(x, y)$$

and

$$v_{G_1}(x, y) \leq \frac{\pi}{\log 3} j_{G_1}(x, y),$$

where the inequalities are sharp if $x = -y$.

3.2. The unit ball $G_2 = \mathbb{B}^n$. For the convenience of geometric explanation, let $x, y \in \mathbb{B}^2$ and $x \neq y$. We define ellipses

$$E_x = \{z \in \mathbb{B}^2 : |x - z| + |z| = 1\}, \quad E_y = \{z \in \mathbb{B}^2 : |y - z| + |z| = 1\}$$

and denote $E_x \cap E_y = \{z_1, z_2\}$ (See Figure 3(a)). We choose z to be that one of the points z_1 and z_2 , which has the larger norm. Then

$$v_{\mathbb{B}^2}(x, y) = \frac{1}{2} \angle(x, z, y).$$

In particular, for $x \neq 0, y = 0$, we have

$$(3.3) \quad v_{\mathbb{B}^2}(0, x) = \arcsin |x| \in (0, \pi/2)$$

and for $|x| = |y| \neq 0, \theta = \frac{1}{2} \angle(x, 0, y) \in (0, \pi/2]$, we have

$$(3.4) \quad v_{\mathbb{B}^2}(x, y) = 2 \arctan \frac{|x| \sin \theta}{1 - |x| \cos \theta}.$$

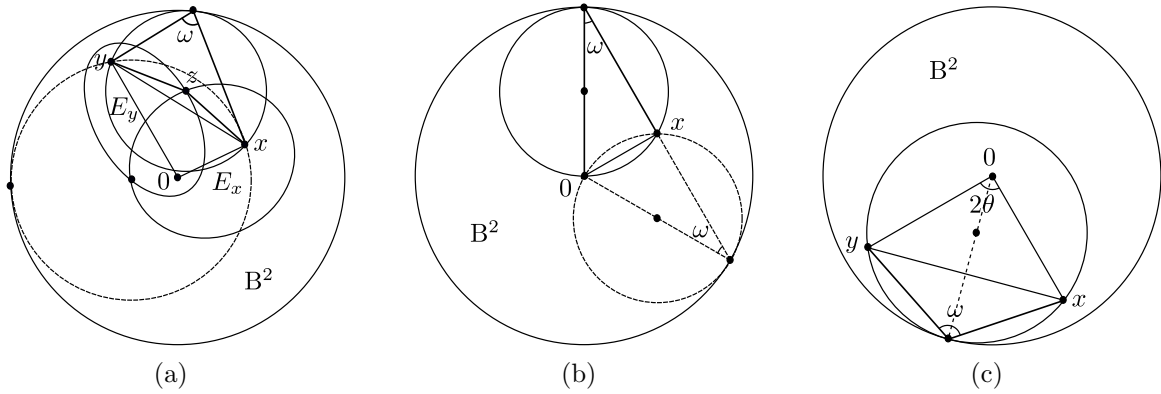


FIGURE 3. The visual angle metric in the unit disk $v_{\mathbb{B}^2}(x, y) = \omega$. (a) General case, where z is in the intersection of ellipses E_x and E_y . (b) Special case (3.3), where $y = 0$. (c) Special case (3.4), where $|x| = |y|$ and $\angle(x, 0, y) = 2\theta$.

For the comparison of the visual angle metric and the hyperbolic metric in the unit ball, we need some technical lemmas.

The so-called *monotone form of l'Hôpital's rule* is useful in deriving monotonicity properties and obtaining inequalities.

Lemma 3.5. [AVV, Theorem 1.25] *Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) . Let $g'(x) \neq 0$ on (a, b) . Then, if $f'(x)/g'(x)$ is increasing(decreasing) on (a, b) , so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

- Lemma 3.6.* (1) The function $f_1(r) \equiv \frac{\arcsin r}{\operatorname{arth} r}$ is strictly decreasing from $(0, 1)$ onto $(0, 1)$.
 (2) The function $f_2(r) \equiv \frac{\arcsin r}{\log(1/(1-r))}$ is strictly decreasing from $(0, 1)$ onto $(0, 1)$.
 (3) The function $f_3(r) \equiv \arctan \frac{cr}{1-c\sqrt{1-r^2}} - \operatorname{arsh} \frac{2cr}{1-c^2}$ is strictly decreasing from $(0, 1)$ onto $(\arctan c - \log \frac{1+c}{1-c}, 0)$ for $c \in (0, 1)$.
 (4) The function $f_4(r) \equiv \frac{\arctan r}{\operatorname{arch}(1+2r^2)}$ is strictly decreasing from $(0, \infty)$ onto $(0, 1/2)$.

Proof. (1) Let $f_{11}(r) = \arcsin r$ and $f_{12}(r) = \operatorname{arth} r$. Then $f_{11}(0^+) = f_{12}(0^+) = 0$. By differentiation,

$$\frac{f'_{11}(r)}{f'_{12}(r)} = \sqrt{1-r^2}$$

which is strictly decreasing on $(0, 1)$. Therefore f_1 is strictly decreasing on $(0, 1)$ by Lemma 3.5. The limiting value $f_1(1^-) = 0$ is clear and $f_1(0^+) = 1$ by l'Hôpital's Rule.

(2) Let $f_{21}(r) = \arcsin r$ and $f_{22}(r) = \log(1/(1-r))$. Then $f_{21}(0^+) = f_{22}(0^+) = 0$. By differentiation,

$$\frac{f'_{21}(r)}{f'_{22}(r)} = \sqrt{\frac{1-r}{1+r}}$$

which is strictly decreasing on $(0, 1)$. Therefore f_2 is strictly decreasing on $(0, 1)$ by Lemma 3.5. The limiting value $f_2(1^-) = 0$ is clear and $f_2(0^+) = 1$ by l'Hôpital's Rule.

(3) Let $r' = \sqrt{1-r^2}$. By differentiation,

$$f'_3(r) = \frac{c}{\sqrt{1+c^2-2cr'}} \left(\frac{r'-c}{r'\sqrt{1+c^2-2cr'}} - \frac{2}{\sqrt{1+c^2+2cr'}} \right).$$

It is clear that $f'_3(r) < 0$ if $r' \leq c$. Therefore we suppose that $r' > c$, namely $0 < r < \sqrt{1-c^2}$ in the sequel. Rewrite

$$f'_3(r) = \frac{2c}{\sqrt{(1+c^2)^2 - (2cr')^2}} \left(\frac{1}{2}\phi(r) - 1 \right),$$

where $\phi(r) = \frac{r'-c}{r'} \sqrt{\frac{1+c^2+2cr'}{1+c^2-2cr'}}$ is strictly decreasing. Therefore, we have $\phi(r) < \phi(0) = 1+c$ and hence $f'_3(r) < 0$ when $r' > c$.

Therefore f_3 is strictly decreasing on $(0, 1)$. The limiting values are clear.

(4) Let $f_{41}(r) = \arctan r$ and $f_{42}(r) = \operatorname{arch}(1+2r^2)$. Then $f_{41}(0^+) = f_{42}(0^+) = 0$. By differentiation,

$$\frac{f'_{41}(r)}{f'_{42}(r)} = \frac{1}{2\sqrt{1+r^2}}$$

which is strictly decreasing on $(0, \infty)$. Therefore f_4 is strictly decreasing by Lemma 3.5. The limiting value $f_4(\infty) = 0$ is clear and $f_4(0^+) = 1/2$ by l'Hôpital's Rule. \square

Lemma 3.7. Let $\alpha \in (0, \pi)$. Then the function

$$f(\theta) = (1 + \cos(\alpha + \theta))(1 + \cos(\alpha - \theta))$$

is strictly decreasing on $(0, \pi - \alpha)$.

Proof. Since $0 < \theta < \pi - \alpha < \pi$, we have $\cos \theta > -\cos \alpha$. Therefore,

$$f'(\theta) = -2 \sin \theta (\cos \theta + \cos \alpha) < 0.$$

Hence $f(\theta)$ is strictly decreasing on $(0, \pi - \alpha)$. \square

Lemma 3.8. *Let $a \in \mathbb{B}^n$, P be any hyperplane through 0 and a . Let $C = S(a, r)$ be a circle centered at a with radius r in $\overline{\mathbb{B}^n} \cap P$ and tangent to S^{n-1} at the point z . Let two distinct points $x', y' \in C$ such that $|x'| = |y'|$ and $\angle(x', z, y') = \alpha \in (0, \pi)$. Then for arbitrary two distinct points $x, y \in C$ with $\angle(x, z, y) = \alpha$, there holds*

$$(1 - |x|^2)(1 - |y|^2) \leq (1 - |x'|^2)(1 - |y'|^2).$$

Proof. Without loss of generality, we may assume that $\overline{\mathbb{B}^n} \cap P = \overline{\mathbb{B}^2}$. Choose two distinct points $x, y \in C$ and $\angle(x, z, y) = \alpha$. By symmetry, we can also assume that $|x| \leq |y|$, and the triples (x, z, y) and (x', z, y') labeled in positive order on C , respectively (see Fig 4).

It is easy to see that $r = 1 - |a|$ and $[x', y'] \perp L(0, z)$, namely x', y' are symmetry with respect to $L(0, z)$. For the inequality, we divide the proof into three cases.

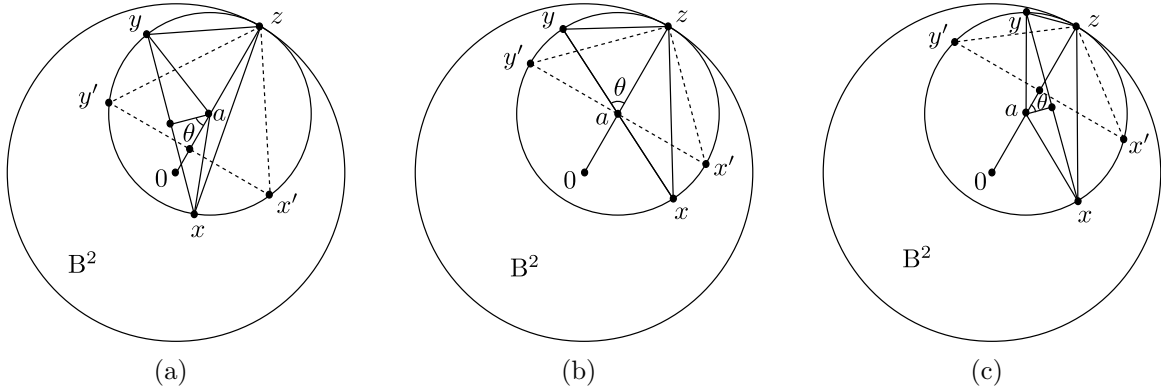


FIGURE 4. Proof of Lemma 3.8. Here $\angle(x, z, y) = \angle(x', z, y') = \alpha$ and $(1 - |x|^2)(1 - |y|^2) \leq (1 - |x'|^2)(1 - |y'|^2)$.

Case 1. $0 < \alpha < \frac{\pi}{2}$.

Let $\theta = \angle(0, a, \frac{x+y}{2}) \in [0, \pi - \alpha]$. It is clear that $\angle(0, a, \frac{x'+y'}{2}) = 0$. Then

$$x = a - \frac{a}{|a|}(1 - |a|)e^{i(\alpha - \theta)} \quad \text{and} \quad y = a - \frac{a}{|a|}(1 - |a|)e^{-i(\alpha + \theta)}.$$

Hence by Lemma 3.7, we have

$$(1 - |x|^2)(1 - |y|^2) = 4|a|^2(1 - |a|)^2 f(\theta) \leq 4|a|^2(1 - |a|)^2 f(0),$$

where $f(\theta) = (1 + \cos(\alpha + \theta))(1 + \cos(\alpha - \theta))$. Namely,

$$(1 - |x|^2)(1 - |y|^2) \leq (1 - |x'|^2)(1 - |y'|^2).$$

Case 2. $\alpha = \frac{\pi}{2}$.

Let $\theta = \angle(z, a, y) \in (0, \pi/2]$. It is clear that $\angle(z, a, x') = \angle(z, a, y') = \frac{\pi}{2}$. Then

$$x = a + \frac{a}{|a|}(1 - |a|)e^{-i(\pi - \theta)} \quad \text{and} \quad y = a + \frac{a}{|a|}(1 - |a|)e^{i\theta}.$$

Hence we have

$$(1 - |x|^2)(1 - |y|^2) = 4|a|^2(1 - |a|)^2 \sin^2 \theta \leq 4|a|^2(1 - |a|)^2 \sin^2 \frac{\pi}{2}.$$

Therefore, we have

$$(1 - |x|^2)(1 - |y|^2) \leq (1 - |x'|^2)(1 - |y'|^2).$$

Case 3. $\frac{\pi}{2} < \alpha < \pi$.

Let $\theta = \angle(z, a, \frac{x+y}{2}) \in [0, \pi - \alpha]$. It is clear that $\angle(z, a, \frac{x'+y'}{2}) = 0$. Then

$$x = a + \frac{a}{|a|}(1 - |a|)e^{-i(\pi - \alpha + \theta)} \quad \text{and} \quad y = a + \frac{a}{|a|}(1 - |a|)e^{i(\pi - \alpha - \theta)}.$$

By a similar argument to Case 1, we have

$$(1 - |x|^2)(1 - |y|^2) = 4|a|^2(1 - |a|)^2 f(\theta) \leq 4|a|^2(1 - |a|)^2 f(0),$$

where $f(\theta) = (1 + \cos(\alpha + \theta))(1 + \cos(\alpha - \theta))$. Namely,

$$(1 - |x|^2)(1 - |y|^2) \leq (1 - |x'|^2)(1 - |y'|^2).$$

By Case 1-3, we complete the proof. □

Corollary 3.9. Let x, y, x', y' be as in Lemma 3.8. Then

$$\rho_{\mathbb{B}^n}(x, y) \geq \rho_{\mathbb{B}^n}(x', y').$$

Lemma 3.10. Let $x \in \mathbb{B}^n$. Then

$$v_{\mathbb{B}^n}(0, x) \leq \rho_{\mathbb{B}^n}(0, x)/2,$$

and

$$v_{\mathbb{B}^n}(0, x) \leq j_{\mathbb{B}^n}(0, x).$$

The constant $\frac{1}{2}$ in the first inequality is best possible and the second inequality is sharp.

Proof. For all $x \in \mathbb{B}^n$ and $x \neq 0$, by (2.9) we have

$$\rho_{\mathbb{B}^n}(0, x) = 2\operatorname{arth}|x| \quad \text{and} \quad j_{\mathbb{B}^n}(0, x) = \log \frac{1}{1 - |x|}.$$

By (3.3), Lemma 3.6(1) and (2), we obtain the inequalities which are sharp if $|x| \rightarrow 0^+$. □

Lemma 3.11. Let $x, y \in \mathbb{B}^n$ and $|x| = |y|$. Then

$$v_{\mathbb{B}^n}(x, y) \leq \rho_{\mathbb{B}^n}(x, y),$$

and the inequality is sharp.

Proof. Let $|x| = |y| \in (0, 1)$ and $\theta = \frac{1}{2}\angle(x, 0, y) \in (0, \pi/2]$. Then by (2.8)

$$\rho_{\mathbb{B}^n}(x, y) = 2\operatorname{arsh} \frac{2|x|\sin\theta}{1 - |x|^2}.$$

By (3.4) and making substitution of $r = \sin\theta$ and $c = |x|$ in Lemma 3.6(3), we have

$$v_{\mathbb{B}^n}(x, y) \leq \rho_{\mathbb{B}^n}(x, y).$$

For the sharpness, let $|x| = |y| = 1 - 1/t$ and $\sin\theta = e^{-t}$ ($t > 0$). Then by l'Hôpital's Rule

$$\lim_{t \rightarrow +\infty} \frac{v_{\mathbb{B}^n}(x, y)}{\rho_{\mathbb{B}^n}(x, y)} = \lim_{t \rightarrow +\infty} \frac{\arctan \frac{1-1/t}{e^{t[1-(1-1/t)\sqrt{1-e^{-2t}}]}}}{\operatorname{arsh} \frac{2(1-1/t)}{e^{t[1-(1-1/t)^2]}}} = \lim_{t \rightarrow +\infty} \frac{1 - (1 - 1/t)^2}{2[1 - (1 - 1/t)\sqrt{1 - e^{-2t}}]} = 1.$$

Together with Lemma 3.10 we obtain the result. □

Theorem 3.12. Let $x, y \in \mathbb{B}^n$. Then

$$v_{\mathbb{B}^n}(x, y) \leq \rho_{\mathbb{B}^n}(x, y),$$

and the inequality is sharp.

Proof. For $x, y \in \mathbb{B}^n$, $x \neq y$ and $|x| \neq |y|$, by Lemma 3.8, Lemma 3.11, Corollary 3.9 there exist $x', y' \in \mathbb{B}^n$ such that $|x'| = |y'|$, $|x - y| = |x' - y'|$ and

$$v_{\mathbb{B}^n}(x, y) = v_{\mathbb{B}^n}(x', y') \leq \rho_{\mathbb{B}^n}(x', y') \leq \rho_{\mathbb{B}^n}(x, y).$$

Together with Lemma 3.11, the inequality holds for all $x, y \in \mathbb{B}^n$ and it is sharp. \square

Conjecture 3.13. There exists a constant $c \in (1.431, 1.432)$ such that for all $x, y \in \mathbb{B}^n$

$$v_{\mathbb{B}^n}(x, y) \leq c j_{\mathbb{B}^n}(x, y).$$

Theorem 3.14. Let $x, y \in \mathbb{B}^n$. Let $\rho_{\mathbb{B}^n}^*(x, y) = \arctan(\operatorname{sh} \frac{\rho_{\mathbb{B}^n}(x, y)}{2})$. Then

$$\rho_{\mathbb{B}^n}^*(x, y) \leq v_{\mathbb{B}^n}(x, y) \leq 2\rho_{\mathbb{B}^n}^*(x, y).$$

Equality holds in the left-hand side if $0, x, y$ are collinear and the constant 2 in the right-hand side of the inequality is best possible.

Proof. It suffices to consider the 2-dimension case.

For $x, y \in \mathbb{B}^2$ and $x \neq y$. Let $z \in E_{xy}^\omega \cap S^1$, where E_{xy}^ω is the envelope such that ω is the supremum angle in Definition 2.21, i.e., $v_{\mathbb{B}^2}(x, y) = \angle(x, z, y) = \omega$. Then there exists exactly one circle $S^1(a, 1 - |a|)$ which is through x, y, z and tangent to S^1 . By Lemma 3.8, there also exist $x', y' \in S^1(a, 1 - |a|)$ such that $\angle(x', z, y') = \angle(x, z, y)$ and $|x'| = |y'|$. For the convenience of proof, we suppose that the triples (x, z, y) and (x', z, y') are labeled in positive order on $S^1(a, 1 - |a|)$, respectively. Without loss of generality, we may assume that $|x| \leq |y|$ (cf. Figure 4). By the proof of Lemma 3.8, we have

$$(1 - |x'|^2)(1 - |y'|^2) = 4|a|^2(1 - |a|)^2(1 + \cos \omega)^2$$

and

$$|x' - y'| = |x - y| = 2(1 - |a|) \sin \omega.$$

Therefore, by Lemma 3.8

$$\begin{aligned} \tan \frac{\omega}{2} &= |a| \frac{|x' - y'|}{\sqrt{(1 - |x'|^2)(1 - |y'|^2)}} \\ (3.15) \quad &\leq \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}} = \operatorname{sh} \frac{\rho_{\mathbb{B}^2}(x, y)}{2}. \end{aligned}$$

Thus we prove the right-hand side of the inequality. For the sharpness, let $x = (1 - t) + i t$ and $y = (1 - t) - i t$ ($0 < t < 1$). Then $x, y \in S^1(1 - t, t)$ and $|x - y| = 2t$. Therefore, we have

$$\lim_{t \rightarrow 0^+} \frac{v_{\mathbb{B}^2}(x, y)}{\rho_{\mathbb{B}^2}^*(x, y)} = \lim_{t \rightarrow 0^+} \frac{\pi}{2} \left(\arctan \frac{1}{1 - t} \right)^{-1} = 2.$$

To prove the left-hand side of the inequality, we only need to consider $v_{\mathbb{B}^n}(x, y) \in (0, \pi/2)$ because $\operatorname{sh} \frac{\rho_{\mathbb{B}^n}(x, y)}{2}$ is always nonnegative. By the proof of Lemma 3.8 Case 1, we also have

$$(1 - |x|^2)(1 - |y|^2) = 4|a|^2(1 - |a|)^2 f(\theta),$$

where $f(\theta) = (1 + \cos(\omega + \theta))(1 + \cos(\omega - \theta))$ and $\theta = \angle(0, a, \frac{x+y}{2})$. It is not difficult to obtain that $f(\theta)$ reaches its minimum if $0, x, y$ are collinear by the definition of the visual angle metric and Lemma 3.7.

Then

$$\begin{aligned} v_{\mathbb{B}^n}(x, y) \geq \rho_{\mathbb{B}^n}^*(x, y) &\Leftrightarrow \tan v_{\mathbb{B}^n}(x, y) \geq \operatorname{sh} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \\ &\Leftrightarrow |a| \sqrt{f(\theta)} \geq \cos \omega. \end{aligned}$$

We suppose that $0, x, y$ are collinear in the sequel and $\alpha = \angle(0, a, \frac{x+y}{2})$ in this case. Let $s = |a|$ and $|\frac{x+y}{2}| = t$.

Case 1. $0 \in B^2(a, 1-s)$.

By the law of cosines

$$\cos(\omega + \alpha) = \frac{s^2 + (1-s)^2 - ((1-s)\sin\omega + t)^2}{2s(1-s)}$$

and

$$\cos(\omega - \alpha) = \frac{s^2 + (1-s)^2 - ((1-s)\sin\omega - t)^2}{2s(1-s)}.$$

Since

$$((1-s)\sin\omega + t)((1-s)\sin\omega - t) = 1 - 2s,$$

we get

$$t^2 = (1-s)^2 \sin^2 \omega - (1-2s).$$

Thus, we have

$$\begin{aligned} 4s^2(1-s)^2 f(\alpha) &= (1 - (1-s)^2 \sin^2 \omega - t^2)^2 - 4t^2(1-s)^2 \sin^2 \omega \\ &= 4(1-s)^2 \cos^2 \omega. \end{aligned}$$

Therefore,

$$s\sqrt{f(\theta)} \geq s\sqrt{f(\alpha)} = \cos\omega.$$

Case 2. $0 \in \mathbb{B}^2 \setminus \bar{B}^2(a, 1-s)$.

By the law of cosines

$$\cos(\omega + \alpha) = \frac{s^2 + (1-s)^2 - (t + (1-s)\sin\omega)^2}{2s(1-s)}$$

and

$$\cos(\omega - \alpha) = \frac{s^2 + (1-s)^2 - (t - (1-s)\sin\omega)^2}{2s(1-s)}.$$

Since

$$(t + (1-s)\sin\omega)(t - (1-s)\sin\omega) = 2s - 1,$$

by the proof of Case 1 we have

$$s\sqrt{f(\theta)} \geq sf(\alpha) = \cos\omega.$$

Case 3. $0 \in S^1(a, 1-s)$.

It is clear that $s = 1/2$ and

$$s\sqrt{f(\theta)} \geq f(\omega)/2 = \cos\omega.$$

Therefore, by Case 1-3 we prove the left-hand side of the inequality, and the equality holds if $0, x, y$ are collinear.

This completes the proof. \square

Corollary 3.16. For all $x, y \in \mathbb{B}^n$, we have

$$v_{\mathbb{B}^n}(x, y) \leq 2 \arctan \frac{|x-y|(2-|x-y|)}{2\sqrt{(1-|x|^2)(1-|y|^2)}},$$

equality holds if $|x| = |y| = (\sqrt{2}\sin(\theta + \frac{\pi}{4}))^{-1}$ and $\theta = \frac{1}{2}\angle(x, 0, y) \in (0, \pi/2)$.

Proof. We still consider the 2-dimension case. Let a be as in the proof of Theorem 3.14. Since $|a| + |a - x| = |a| + |a - y| = 1$, we get $|a| = \frac{2-(|a-x|+|a-y|)}{2} \leq \frac{2-|x-y|}{2}$, then by (3.15) we prove the inequality.

For the equality, let $|x| = |y| > 0$ and $\theta = \frac{1}{2}\angle(x, 0, y) > 0$, then

$$(3.17) \quad \frac{|x - y|(2 - |x - y|)}{2\sqrt{(1 - |x|^2)(1 - |y|^2)}} = \frac{2|x| \sin \theta (1 - |x| \sin \theta)}{1 - |x|^2}.$$

By (3.4) and (3.17) the equality holds if $|x| = |y| = \frac{1}{\sin \theta + \cos \theta}$. \square

3.18. The upper half space $G_3 = \mathbb{H}^n$. For G_3 it is sufficient to consider the case $n = 2$. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{H}^2$ and $x \neq y$. Then the circle through x, y and tangent to $\partial\mathbb{H}^2$ with center

$$z = \frac{x_1 y_2 - x_2 y_1 + \sqrt{x_2 y_2} |x - y|}{y_2 - x_2} + i \frac{(x_2 + y_2) |x - y|^2 + 2\sqrt{x_2 y_2} (x_1 - y_1) |x - y|}{2(y_2 - x_2)^2}$$

or

$$z' = \frac{x_1 y_2 - x_2 y_1 - \sqrt{x_2 y_2} |x - y|}{y_2 - x_2} + i \frac{(x_2 + y_2) |x - y|^2 - 2\sqrt{x_2 y_2} (x_1 - y_1) |x - y|}{2(y_2 - x_2)^2}$$

if $x_2 \neq y_2$ and

$$w = \frac{x_1 + y_1}{2} + i \frac{4x_2^2 + (x_1 - y_1)^2}{8x_2}$$

if $x_2 = y_2$ (see Figure 5).

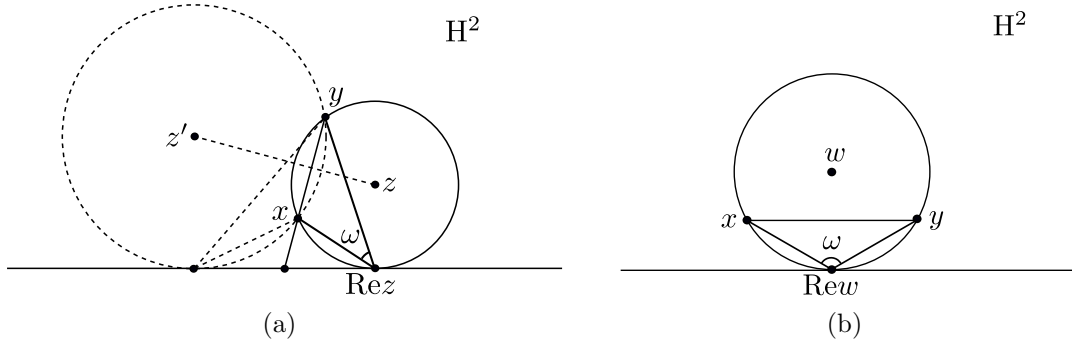


FIGURE 5. The visual angle metric in the upper half plane $v_{\mathbb{H}^2}(x, y) = \omega$.

Therefore,

$$(3.19) \quad v_{\mathbb{H}^2}(x, y) = \begin{cases} \arccos \frac{2\sqrt{x_2 y_2} |x - y| + (x_1 - y_1)(x_2 + y_2)}{(x_2 + y_2) |x - y| + 2\sqrt{x_2 y_2} (x_1 - y_1)}, & x_1 \leq y_1, x_2 < y_2, \\ \arccos \frac{4x_2^2 - (x_1 - y_1)^2}{4x_2^2 + (x_1 - y_1)^2}, & x_2 = y_2. \end{cases}$$

In particular, if $x_2 = y_2$ and $y_1 = -x_1 > 0$, then

$$(3.20) \quad v_{\mathbb{H}^2}(x, y) = 2 \arctan \frac{y_1}{y_2}.$$

If $x_1 = y_1$, then

$$(3.21) \quad v_{\mathbb{H}^2}(x, y) = \arccos \frac{2\sqrt{x_2 y_2}}{x_2 + y_2}.$$

Lemma 3.22. *Let $a \in \mathbb{H}^n$. Let $C = S(a, r)$ be a circle centered at a with radius r in $\overline{\mathbb{H}}^n$ and tangent to $\partial\mathbb{H}^n$ at point z . Let two distinct points $x', y' \in C$ such that $|x' - z| = |y' - z|$ and $\angle(x', z, y') = \alpha \in (0, \pi)$. Then for arbitrary two distinct points $x, y \in C$ with $\angle(x, z, y) = \alpha$, there holds*

$$(3.23) \quad d(x, \partial\mathbb{H}^n)d(y, \partial\mathbb{H}^n) \leq d(x', \partial\mathbb{H}^n)d(y', \partial\mathbb{H}^n).$$

Proof. Without loss of generality, we may assume that $C \in \overline{\mathbb{H}}^2$ and $z = 0$. Choose two distinct points $x, y \in C$ and $\angle(x, z, y) = \alpha$. By symmetry, we can also assume that $|x| \leq |y|$, and the triples (x, z, y) and (x', z, y') are labeled in positive order on C , respectively (see Figure 6).

It is clear that $r = |a|$, $[x', y'] \perp L(0, a)$, namely x', y' are symmetry with respect to $L(0, a)$. Furthermore, inequality (3.23) reduces to

$$\operatorname{Im} x \operatorname{Im} y \leq \operatorname{Im} x' \operatorname{Im} y'.$$

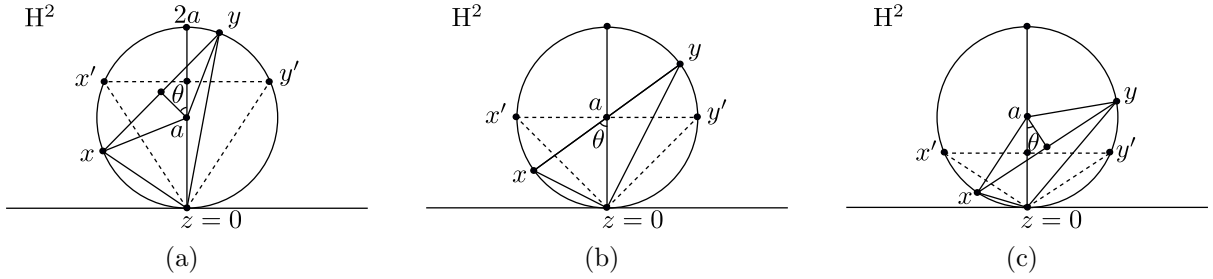


FIGURE 6. Proof of Lemma 3.22. Here $\angle(x, z, y) = \angle(x', z, y') = \alpha$ and $\operatorname{Im} x \operatorname{Im} y \leq \operatorname{Im} x' \operatorname{Im} y'$.

In the same way as in the proof of Lemma 3.8, we also divide the proof into three cases.

Case 1. $0 < \alpha < \frac{\pi}{2}$.

Let $\theta = \angle(2a, a, \frac{x+y}{2}) \in [0, \pi - \alpha)$. It is clear that $\angle(2a, a, \frac{x'+y'}{2}) = 0$. Then

$$x = a(1 + e^{i(\alpha+\theta)}) \quad \text{and} \quad y = a(1 + e^{-i(\alpha-\theta)}).$$

Moreover,

$$\operatorname{Im} x \operatorname{Im} y = |a|^2 f(\theta),$$

where $f(\theta) = (1 + \cos(\alpha + \theta))(1 + \cos(\alpha - \theta))$, by Lemma 3.7, we have

$$\operatorname{Im} x \operatorname{Im} y \leq \operatorname{Im} x' \operatorname{Im} y'.$$

Case 2. $\alpha = \frac{\pi}{2}$.

Let $\theta = \angle(0, a, x) \in (0, \pi/2]$. It is clear that $\angle(0, a, x') = \angle(0, a, y') = \frac{\pi}{2}$. Then

$$x = a(1 - e^{-i\theta}) \quad \text{and} \quad y = a(1 - e^{i(\pi-\theta)}).$$

Moreover,

$$\operatorname{Im} x \operatorname{Im} y = |a|^2 \sin^2 \theta \leq |a|^2 \sin^2 \frac{\pi}{2},$$

and hence

$$\operatorname{Im} x \operatorname{Im} y \leq \operatorname{Im} x' \operatorname{Im} y'.$$

Case 3. $\frac{\pi}{2} < \alpha < \pi$.

Let $\theta = \angle(0, a, \frac{x+y}{2}) \in [0, \pi - \alpha)$. It is clear that $\angle(0, a, \frac{x'+y'}{2}) = 0$. Then

$$x = a(1 - e^{-i(\pi-\alpha-\theta)}) \quad \text{and} \quad y = a(1 - e^{i(\pi-\alpha+\theta)}).$$

Moreover,

$$\operatorname{Im} x \operatorname{Im} y = |a|^2 f(\theta),$$

where $f(\theta) = (1 + \cos(\alpha + \theta))(1 + \cos(\alpha - \theta))$, by Lemma 3.7 we have

$$\operatorname{Im} x \operatorname{Im} y \leq \operatorname{Im} x' \operatorname{Im} y'.$$

By Case 1-3, we complete the proof. \square

Corollary 3.24. Let x, y, x', y' be as in Lemma 3.22. Then

$$\rho_{\mathbb{H}^n}(x, y) \geq \rho_{\mathbb{H}^n}(x', y').$$

Lemma 3.25. Let $x, y \in \mathbb{H}^n$ and $d(x, \partial\mathbb{H}^n) = d(y, \partial\mathbb{H}^n)$. Then

$$v_{\mathbb{H}^n}(x, y) \leq \rho_{\mathbb{H}^n}(x, y),$$

and the inequality is sharp.

Proof. It suffices to prove for the 2-dimension case. Let x, y be two distinct points in \mathbb{H}^2 . Since both metrics are invariant under translations, we may assume that $y_1 = \operatorname{Re} y = -\operatorname{Re} x > 0$ and $y_2 = \operatorname{Im} y = \operatorname{Im} x > 0$. Then

$$\rho_{\mathbb{H}^2}(x, y) = \operatorname{arch}\left(1 + 2\left(\frac{y_1}{y_2}\right)^2\right).$$

By (3.20) and making substitution of $r = \frac{y_1}{y_2}$ in Lemma 3.6(4), we have

$$v_{\mathbb{H}^2}(x, y) \leq \rho_{\mathbb{H}^2}(x, y)$$

and the inequality is sharp if $r \rightarrow 0^+$. \square

Theorem 3.26. Let $x, y \in \mathbb{H}^n$. Then

$$v_{\mathbb{H}^n}(x, y) \leq \rho_{\mathbb{H}^n}(x, y),$$

and the inequality is sharp.

Proof. For $x, y \in \mathbb{H}^n$, $x \neq y$ and $d(x, \partial\mathbb{H}^n) \neq d(y, \partial\mathbb{H}^n)$, by Lemma 3.22, Lemma 3.25 and Corollary 3.24, there exist $x', y' \in \mathbb{H}^n$ such that $d(x', \partial\mathbb{H}^n) = d(y', \partial\mathbb{H}^n)$, $|x - y| = |x' - y'|$ and

$$v_{\mathbb{H}^n}(x, y) = v_{\mathbb{H}^n}(x', y') \leq \rho_{\mathbb{H}^n}(x', y') \leq \rho_{\mathbb{H}^n}(x, y).$$

Together with Lemma 3.25, the inequality holds for all $x, y \in \mathbb{H}^n$ and it is sharp. \square

Conjecture 3.27. There exists a constant $c \in (1.432, 1.433)$ such that for all $x, y \in \mathbb{H}^n$

$$v_{\mathbb{H}^n}(x, y) \leq c j_{\mathbb{H}^n}(x, y).$$

Theorem 3.28. Let $x, y \in \mathbb{H}^n$. Let $\rho_{\mathbb{H}^n}^*(x, y) = \arctan(\operatorname{sh} \frac{\rho_{\mathbb{H}^n}(x, y)}{2})$. Then

$$\rho_{\mathbb{H}^n}^*(x, y) \leq v_{\mathbb{H}^n}(x, y) \leq 2\rho_{\mathbb{H}^n}^*(x, y).$$

Equality holds in the left-hand side if $L(x, y - x)$ is perpendicular to the boundary and in the right-hand side if $L(x, y - x)$ is parallel to the boundary.

Proof. It suffices to consider the 2-dimension case.

For $x, y \in \mathbb{H}^2$ and $x \neq y$. Let $z \in E_{xy}^\omega \cap \partial\mathbb{H}^2$, where E_{xy}^ω is the envelope such that ω is the supremum angle in Definition 2.21, i.e., $v_{\mathbb{H}^2}(x, y) = \angle(x, z, y) = \omega$. Then there exists exactly one circle $S^1(a, r)$ which is through x, y, z and tangent to $\partial\mathbb{H}^2$. By Lemma 3.22, there also exist $x', y' \in S^1(a, r)$ such that $\angle(x', z, y') = \angle(x, z, y)$ and $\operatorname{Im} x' = \operatorname{Im} y'$. For the convenience of proof, we suppose that the triples (x, z, y) and (x', z, y') labeled in

positive order on $S^1(a, r)$, respectively. Without loss of generality, we may assume that $z = 0$ and $|x| \leq |y|$ (cf. Figure 6). By the proof of Lemma 3.22, we have

$$\operatorname{Im} x' \operatorname{Im} y' = |a|^2(1 + \cos \omega)^2$$

and

$$|x' - y'| = |x - y| = 2|a| \sin \omega.$$

Therefore, by Lemma 3.22

$$\tan \frac{\omega}{2} = \frac{|x' - y'|}{2\sqrt{(\operatorname{Im} x' \operatorname{Im} y')}} \leq \frac{|x - y|}{2\sqrt{(\operatorname{Im} x \operatorname{Im} y)}} = \sqrt{\frac{1}{2}(\operatorname{ch} \rho_{\mathbb{H}^2}(x, y) - 1)} = \operatorname{sh} \frac{\rho_{\mathbb{H}^2}(x, y)}{2},$$

which implies the right-hand side of the inequality with equality if $\operatorname{Im} x = \operatorname{Im} y$.

To prove the left-hand side of the inequality, we only need to consider $v_{\mathbb{H}^2}(x, y) \in (0, \pi/2)$ since $\operatorname{sh} \frac{\rho_{\mathbb{H}^2}(x, y)}{2}$ is always nonnegative. By the proof of Case 1 in Lemma 3.22, we have

$$\operatorname{Im} x \operatorname{Im} y = |a|^2 f(\theta),$$

where $f(\theta) = (1 + \cos(\omega + \theta))(1 + \cos(\omega - \theta))$ and $\theta = \angle(2a, a, \frac{x+y}{2}) \in [0, \pi/2]$ by the definition of the visual angle metric. Then

$$\begin{aligned} v_{\mathbb{H}^2}(x, y) \geq \rho_{\mathbb{H}^2}^*(x, y) &\Leftrightarrow \tan v_{\mathbb{H}^2}(x, y) \geq \operatorname{sh} \frac{\rho_{\mathbb{H}^2}(x, y)}{2} \\ &\Leftrightarrow \sqrt{f(\theta)} \geq \cos \omega. \end{aligned}$$

Since $\pi/2 \in [0, \pi - \omega]$, by Lemma 3.7 we have

$$f(\theta) \geq f(\pi/2) = \cos^2 \omega.$$

Thus we prove the left-hand side of the inequality, and equality holds if $\operatorname{Re} x = \operatorname{Re} y$.

This completes the proof. \square

Proposition 3.29. For $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$. Then $\rho_G^*(x, y) = \arctan(\operatorname{sh} \frac{\rho_G(x, y)}{2})$ is a Möbius-invariant metric.

Proof. The function $f : x \mapsto \arctan(\operatorname{sh}(x/2))$ is increasing on $[0, \infty)$, $f(x)/x$ is decreasing $(0, \infty)$, and $f(0) = 0$. Therefore, by Lemma 2.29 $\rho_G^*(x, y)$ is a metric, and the Möbius invariance follows immediately by the Möbius invariance of $\rho_G(x, y)$. \square

Proof of Theorem 1.1. By Theorem 3.14 and Theorem 3.28, the results follow immediately. \square

4. LIPSCHITZ CONSTANTS UNDER MÖBIUS TRANSFORMATIONS

It is clear that the visual angle metric is similarity invariant but not Möbius invariant. However, this metric is not changed by more than a factor 2 under the Möbius transformations from the unit ball onto itself and onto the upper half space, and from the upper half space onto itself by Theorem 1.1 and Proposition 3.29. In this section, we prove the main theorems of sharp Lipschitz constants for the visual angle metric under several Möbius transformations.

Proof of Theorem 1.2. It suffices to prove the 2-dimension case by the definition of the visual angle metric. By Theorem 1.1 and the Möbius invariance of $\rho_{\mathbb{B}^2}^*(x, y)$, it is clear that

$$v_{\mathbb{B}^2}(x, y)/2 \leq v_{\mathbb{B}^2}(f(x), (y)) \leq 2v_{\mathbb{B}^2}(x, y).$$

For the sharpness, let $a \in (0, 1)$, then $T_a(z) = \frac{a-z}{1-\bar{a}z} \in \mathcal{GM}(\mathbb{B}^2)$. Let $x = it$ and $y = -it$ ($0 < t < 1$). Then

$$T_a(x) = \frac{a(1+t^2) - it(1-a^2)}{1+a^2t^2} \quad \text{and} \quad T_a(y) = \frac{a(1+t^2) + it(1-a^2)}{1+a^2t^2}.$$

Since $|x| = |y|$ and $|T_a(x)| = |T_a(y)|$, by (3.4) we have

$$\lim_{a \rightarrow 1^-} \lim_{t \rightarrow 1^-} \frac{v_{\mathbb{B}^2}(T_a(x), T_a(y))}{v_{\mathbb{B}^2}(x, y)} = \lim_{a \rightarrow 1^-} \lim_{t \rightarrow 1^-} \frac{\arctan \frac{t(1+a)}{1-at^2}}{\arctan t} = \lim_{a \rightarrow 1^-} \frac{4}{\pi} \arctan \frac{1+a}{1-a} = 2.$$

This completes the proof of Theorem 1.2. \square

Conjecture 4.1. *Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$ be a Möbius transformation with $f(a) = 0$. Then*

$$\sup_{x \neq y \in \mathbb{B}^n} \frac{v_{\mathbb{B}^n}(f(x), f(y))}{v_{\mathbb{B}^n}(x, y)} = \frac{4}{\pi} \arctan \frac{1+|a|}{1-|a|}.$$

Proof of Theorem 1.3. By Theorem 1.1 and Proposition 3.29, the inequality is clear.

Without loss of generality, we may assume that the Möbius transformation f maps $a \in \mathbb{H}^2$ to 0. Then f is of the form

$$f(z) = e^{i\alpha} \frac{z-a}{z-\bar{a}}$$

and hence

$$f^{-1}(z) = \frac{a - \bar{a}e^{-i\alpha}z}{1 - e^{-i\alpha}z},$$

where α is a real constant. Since the visual angle metric is invariant under translations, stretchings of \mathbb{H}^2 onto itself and rotations of \mathbb{B}^2 onto itself, we can assume that $a = i$ and $\alpha = 0$. Then we have

$$f(z) = \frac{z-i}{z+i} \quad \text{and} \quad f^{-1}(z) = i \frac{1+z}{1-z}.$$

For the sharpness of the upper bound, let $x = -\frac{2t}{\sqrt{1-t^2}} + i$ and $y = i\frac{1+t}{1-t}$ ($0 < t < 1$). Then

$$f(x) = t^2 + it\sqrt{1-t^2} \quad \text{and} \quad f(y) = t.$$

It is easy to see that $|f(x)| = |f(y)| = t$ and $f(x) \in S^1(1/2, 1/2)$. Hence $\cos \frac{\angle(f(x), 0, f(y))}{2} = \sqrt{(1+t)/2}$ and $\sin(\angle(f(x), 0, f(y))/2) = \sqrt{(1-t)/2}$. By (3.4), we have

$$\begin{aligned} \lim_{t \rightarrow 1^-} v_{\mathbb{B}^2}(f(x), f(y)) &= \lim_{t \rightarrow 1^-} 2 \arctan \frac{t\sqrt{1-t}}{\sqrt{2}-t\sqrt{1+t}} \\ (4.2) \quad &= 2 \arctan \lim_{t \rightarrow 1^-} \frac{\sqrt{1+t}}{\sqrt{1-t}(3t+2)} = \pi. \end{aligned}$$

By (3.19), we have

$$(4.3) \quad \lim_{t \rightarrow 1^-} v_{\mathbb{H}^2}(x, y) = \lim_{t \rightarrow 1^-} \arccos \frac{\sqrt{2}\sqrt{1-t^2} - \sqrt{1-t}}{\sqrt{2} - \sqrt{1-t^2}\sqrt{1-t}} = \frac{\pi}{2}.$$

Therefore, by (4.2) and (4.3), we get the upper bound.

For the sharpness of the lower bound, let $x = 0$ and $y = \frac{t^2}{t^2+4} - i\frac{2t}{t^2+4}$ ($t > 0$). Then

$$f^{-1}(x) = i \quad \text{and} \quad f^{-1}(y) = t + i.$$

By (3.3) and (3.19), we have

$$v_{\mathbb{H}^2}(f^{-1}(x), f^{-1}(y)) = \arccos \frac{4 - t^2}{4 + t^2} \quad \text{and} \quad v_{\mathbb{B}^2}(x, y) = \arcsin \frac{t}{\sqrt{t^2 + 4}}.$$

Since

$$\cos(v_{\mathbb{H}^2}(f^{-1}(x), f^{-1}(y))) = 1 - 2\sin^2(v_{\mathbb{B}^2}(x, y)),$$

we get

$$v_{\mathbb{H}^2}(f^{-1}(x), f^{-1}(y)) = 2v_{\mathbb{B}^2}(x, y).$$

This completes the proof of Theorem 1.3. \square

Proof of Theorem 1.4. By Theorem 1.1 and the Möbius invariance of $\rho_{\mathbb{H}^2}^*(x, y)$, it is clear that

$$v_{\mathbb{H}^2}(x, y)/2 \leq v_{\mathbb{H}^2}(f(x), f(y)) \leq 2v_{\mathbb{H}^2}(x, y).$$

For the sharpness, we divide the proof into two cases.

Case 1. $c \neq 0$ and $d \neq 0$.

Let $x = i$ and $y = i\frac{d^2}{c^2}$. Then

$$f(x) = \frac{ac + bd}{c^2 + d^2} + i\frac{1}{c^2 + d^2} \quad \text{and} \quad f(y) = \frac{ad^3 + bc^3}{cd^3 + c^3d} + i\frac{1}{c^2 + d^2}.$$

Since $\operatorname{Re} x = \operatorname{Re} y = 0$ and $\operatorname{Im} f(x) = \operatorname{Im} f(y) = \frac{1}{c^2 + d^2}$, by Theorem 3.28 and Proposition 3.29 we have

$$\frac{v_{\mathbb{H}^2}(f(x), f(y))}{v_{\mathbb{H}^2}(x, y)} = \frac{2\rho_{\mathbb{H}^2}^*(f(x), f(y))}{\rho_{\mathbb{H}^2}^*(x, y)} = 2.$$

Case 2. $c \neq 0$ and $d = 0$.

Then $bc = -1$ and $f(z) = -\frac{b^2}{z} - ab$. It suffices to consider the map $f(z) = -\frac{1}{z}$ since the visual angle metric is invariant under translations and stretchings from the upper half plane onto itself.

Let $x = te^{i(\pi-t)}$ and $y = i\frac{t}{\sin t}$ ($0 < t < \pi/2$). Then

$$f(x) = \frac{\cos t}{t} + i\frac{\sin t}{t} \quad \text{and} \quad f(y) = i\frac{\sin t}{t}.$$

Since $\operatorname{Im} f(x) = \operatorname{Im} f(y)$, by (3.19) we have

$$(4.4) \quad \lim_{t \rightarrow 0^+} v_{\mathbb{H}^2}(f(x), f(y)) = \lim_{t \rightarrow 0^+} \arccos \frac{4\sin^2 t - \cos^2 t}{4\sin^2 t + \cos^2 t} = \pi$$

and by (3.19)

$$(4.5) \quad \lim_{t \rightarrow 0^+} v_{\mathbb{H}^2}(x, y) = \lim_{t \rightarrow 0^+} \arccos \sin t = \frac{\pi}{2}.$$

Therefore, by (4.4) and (4.5), we get

$$\lim_{t \rightarrow 0^+} \frac{v_{\mathbb{H}^2}(f(x), f(y))}{v_{\mathbb{H}^2}(x, y)} = 2.$$

This completes the proof of Theorem 1.4. \square

Remark 4.6. If $c = 0$ in Theorem 1.4, then $f(z) = a^2z + ab$. Therefore, it is clear that the Lipschitz constant under f for the visual angle metric is always 1.

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